Guy Bomford Prize Lecture
Advancing the theoretical apparatus of physical geodesy

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Mister President, Mister Secretary General, Ladies and Gentlemen, I am very pleased and honoured. The other day, I looked at the list of the Guy Bomford Prize awardees. Honestly, I feel like a hobbit who has just arrived in the Rivendell and there are the wise Elves all around him. I will express my gratitude at the end of this talk. Now, I would like to tell you a few words about the advancements of the theoretical apparatus of physical geodesy. This is the topic, which I have been very passionate about and also quite productive in terms of research outputs, and brought me here on this stage tonight. Vast majority of this research has been performed at the University of West Bohemia in the Czech Republic together with my colleagues *Pavel Novák* and *Martin Pitoňák*. Currently, however, I am employed at the University of Newcastle in Australia.

You certainly know that gravitational field reveals important properties of our planet and it definitely deserves to be one of the main pillars of geodesy. Enormous amount of gravitational data has been collected by various sensors in the past decades. This has mainly stimulated for various revolutionary applications in geosciences. But those who are more theoretically gifted have also been working hard. They have improved the existing concepts and have even gone way further than allowed by the available datasets. In my opinion, we have experienced a renaissance in the theoretical developments in the past few years. My colleagues and I have contributed to these advancements by solving interesting theoretical problems. For simplicity, I divided our research contributions into three categories and it is now my intention to briefly describe each of them.

The first group is about a more complete picture of the third-order gravitational tensor. I came with this idea more or less out of curiosity. No wonder that I encountered strong resistance and doubts. Fortunately, I could find papers and patents reporting actual measurements of the third-order potential derivatives that saved my initial goal. On the left-hand side (see Fig. 1), you can see the mathematical expression of the third-order gravitational tensor in the gradient and component forms. On the right-hand side, you can observe its visualisation as a cube with 27 components. Apparently, the third-order gravitational tensor is a complex mathematical object.

$$\mathbf{T}^{(3)} = \nabla\otimes\nabla\otimes\nabla\, T = \sum_{o,p,q} T_{opq}\, \mathbf{e}_o\otimes\mathbf{e}_p\otimes\mathbf{e}_q\,, \quad o,p,q \in \{x,y,z\}$$

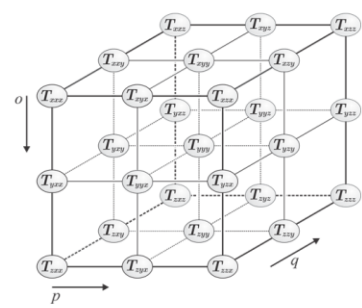

Figure 1: The third-order gravitational tensor

We discovered the secrets of this tensor by looking at its basic properties, such as symmetries of the components, and by constructing the differential operators (Šprlák and Novák 2015). We then found how the gravitational potential could be determined from the third-order tensor components by solving the spherical gravitational curvature boundary-value problem (Šprlák and Novák 2016). We continued further with harmonic analysis and even investigated what could be achieved when observing the third-order potential derivatives by a satellite (Šprlák et al. 2016). Later on, my younger colleagues dealt with more practical aspects, such as non-singular harmonic synthesis (Hamáčková et al. 2016)

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or the integral inversion of these quantities (Pitoňák et al. 2017). In this way, we collected necessary bits and pieces and built the first systematic methodology for the third-order gravitational tensor, just like it had been done decades ago, but for lower-rank gravitational tensors.

The second group of theoretical problems are those on integral transformations. An excellent example of an integral transformation is the famous Stokes integral, which every geodesist and even surveyor knows. It nicely illustrates how gravity anomalies are converted to geoid undulations. It also represents analytical solution of the third boundary-value problem. For real world applications, we directly calculate this integral or solve its inverse.

There are many quantities that we would like to transform from one to another (see Fig. 2). These are the disturbing potential on the left, and its first, second, and third-order derivatives towards right. The quantities that we integrate over are on the lower level at the reference sphere of radius R. The computed quantities are at the upper level at the radius r. Imaginary line connecting one box from the lower level with one box at the upper level could define an existing integral transformation.

Many geodesists have added lines to this schematic and so have we in a collection of seven research papers published in Journal of Geodesy. These publications are listed in the box on the left (see Fig. 2). The different colours help us to identify individual lines in the diagram and thus the related quantities. In total, we derived impressive 98 integral transformations. But we were not obsessed only with equations. We investigated spatial behaviour of the integral kernels that leads to some practical implications, such as efficient numerical calculation, the effect of the distant zones, or suitability for inverse problems. Also, almost every integral formula was implemented in a computer program and validated in a closed-loop simulation.

Finally, our effort culminated in a review paper published in Earth-Science Reviews (Novák et al. 2017). Here, we summarised mutual integral formulas among the disturbing gravitational potential, its first, second, and third-order derivatives including eventually all references from the geodetic literature. Thus, the diagram in Fig. 2 has been completed.

So far, we have silently assumed spherical geometry. Once you solve numerous tasks using this approximation, you get a little bored and the next challenging step is to consider its spheroidal equivalent. This is the third part of my talk. We use a spheroid instead of a sphere, because it more closely fits the shape and the gravitational field of our planet. On the other hand, we lose the comfort of the azimuthal symmetry. In other words, every direction is somewhat different on the spheroidal surface and we cannot simply introduce the polar coordinates that are so efficient on the sphere. In addition, spheroidal counterparts of the addition theorem and orthogonality relationships either do not exist or are defined differently from the familiar spherical case.

With the help of great findings by geodetic forefathers, we could handle these complications and expand the existing class of spheroidal integral transformations. Firstly, we formulated a mathematical model for inverting gravimetric and gradiometric measurements to the gravitational potential (Novák and Šprlák 2018). Secondly, we applied two orthogonalisation approaches to solve the spheroidal vertical and horizontal boundary-value problems (Šprlák and Tangamrongsub 2018). There are other interesting tasks to be solved, thus the spheroidal story is not over yet.

Because we addressed all three parts, it is now time to conclude. You may have seen that we have significantly extended the theoretical apparatus of physical geodesy. These advancements are not restricted to geodesy and also enhance the more general framework of the potential theory. The new mathematical formulations represent the basis for the gravitational field modelling and we have applied some of them in several applied studies. We are confident that the new complex mathematical models can be implemented and describe the reality very well. The presented work has been a part of activities within the Joint Study Group called "Integral equations of potential theory for continuation and transformation of classical and new gravitational observables" of the Inter-Commission Committee on Theory under the umbrella of the International Association of Geodesy. For the first time, I have acted as a chairman of this study group that has been a very pleasant experience in addition to deriving dozens of equations.

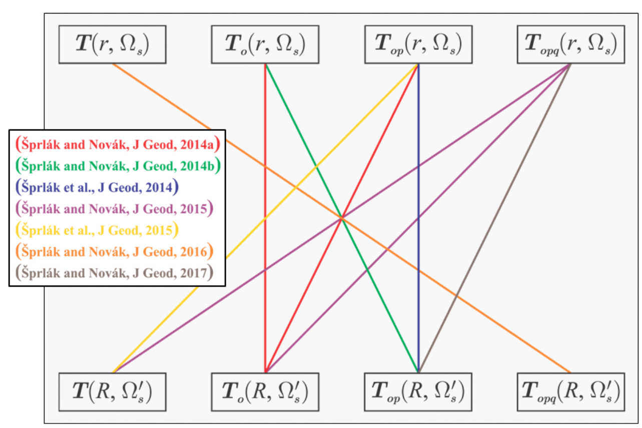

Figure 2: Schematic of the derived integral transformations

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